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Mathematics and the Arts: The Beauty Behind the Math

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Reading the title of this paper, you might ask yourself, "What does the study of mathematics have to do with art?" Furthermore, "How would something so analytical and scientific be related to an entire form of study based on creativity and the abstract?" I ask myself the same questions still, but within mathematics there is a beauty of elegance, of simple proofs, and so on. I will split the topic into two sections for this paper: the first deals with art and beauty found in mathematics. The second is the mathematics found in all forms of what can be considered art. The definition of art is difficult to summarize, so I thought it would be only reasonable to include mathematics in the visual and architectural arts.

I want to start this essay with a quote by Bertrand Russell, a British polymath and logician known for his contributions in set theory, linguistics, and artificial intelligence. In his book *Mysticism and Logic*, he encapsulates the beauty of math perfectly.

> Mathematics, rightly viewed, possesses not only truth but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as poetry. [**9**, p. 48]

Beauty in mathematics itself can be divided into different areas. These include the beauty in technique, method, pattern, and results. For this paper, I will primarily focus on well-known topics such as the golden ratio and the Fibonacci sequence. I will give examples of them in art ranging from that of the ancient Egyptians to the renaissance age to the modern-day. However, not all mathematics is as complex and intricate as the golden ratio and the Fibonacci sequence. Before diving into both of those, let me start with the simple connections between math and art.

Perspective, proportion, symmetry, and lines are all standard terms and techniques used when composing art, but they are also mathematical. This is not limited to modern art; art from all eras involves mathematics. A famous example of proportion in historic art would be Polykleitos's *Doryphoros*, a sculpture depicting a muscular warrior. In *The Canon Of Polykleitos: A Question Of Evidence*, Andrew Stewart states, "The system adopted appears to have taken the form of a series of ratios, which related all the parts of the body proportionally to each other and to the whole…" [**10**, p. 126]. The ratio mentioned does not refer to the golden ratio, but rather to the ratio that signifies "perfect" body proportions according to Polykleitos, which was roughly around 1.14142 [**11**, p. 308] His influence on Roman, Greek, and Renaissance sculpture was immeasurable and his sculptures, while none survived, expressed mathematical beauty and physical perfection in the male physique.

Perspective is another common mathematical concept used in art. Of course, the composition would look wonky and unnatural if the perspective was out of control and chaotic. Perspective is essentially trying to depict a three-dimensional object on a two-dimensional canvas; it is a way to show depth. Algebra can be used to determine the correct position of the object at hand according to the different viewing points or perspectives. Perspective is an art concept that is heavily influenced by mathematics. However, there are other important concepts that I would like to focus on in this paper.

Considered the most natural and beautiful, the golden proportion, also known as the divine ratio in the Renaissance and currently known as the golden ratio, is a concept that was first studied by Pythagoras (569-500 B.C.E) and the Pythagorean school for its aesthetic value. To help define the golden ratio, let's visualize it. Let a line AB of length (l) be divided into two segments by point C, where C splits the segment unequally. Let the lengths AC and CB be a and b, respectively. If C is a point such that l:a as a:b, which is equivalent to $\frac{a+b}{a} = \frac{a}{b}$, then C is the "golden cut" or the golden section of AB. [**4**, p. 25]

The ratio of $\frac{l}{a}$ or $\frac{a}{b}$ is known as the golden ratio and is denoted by phi (φ).

*Figure SEQ Figure * ARABIC 1 Line segment DB* [**4**, p. 25]

We can calculate the numerical value of the golden ratio. let AC = x, CB = 1, so that AC/CB = x = φ. We get:

$$\frac{x+1}{x} = \frac{x}{1}, \text{ i.e., } x^2 - x - 1 = 0$$

$$\text{The positive solution is } x = \frac{1+\sqrt{5}}{2} \approx 1.61803$$

Check [**4**, pp.25-26] for further information on the reciprocal of Phi (φ) and the negative solution (φ') of the above equation.

One cannot describe the golden ratio without mentioning the Fibonacci sequence. It may not seem obvious, but the Fibonacci sequence has much to do with art composition. It is also closely related to the Golden Ratio as the sequence of ratios of consecutive Fibonacci numbers   converges onto Phi (φ).

$$\frac{f_n}{f_{n-1}} \to \varphi$$

To help define and visualize the Fibonacci sequence, I'll use the explanation given in David M. Burton's *History of Math: An Introduction*. Fibonacci originally posed this question about a rabbit pair's offspring:

> A man put one pair of rabbits in a certain place entirely surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year, if the nature of these rabbits is such that every month each

pair bears a new pair which from the second month on becomes productive? [**1**, p. 287]

On the basis that none of the rabbits die, a pair is born during the first month so that there are two pairs present. During the second month, the original pair has produced another pair. One month later, both the original pair and the firstborn pair have produced new pairs, so that two adult and three young pairs are present, and so on. It is important to observe that the young pairs grow up each month and become adult pairs, making the new "adult" entry the previous one plus the previous "young" entry. Each of the pairs that were adults last month produces one young pair, so that the new young entry is equal to the previous adult entry. [**1**, pp. 287 - 288] So, when continued indefinitely, the sequence looks like this:

1, 1, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, … and so on.

This specific pattern of numbers is known as the Fibonacci sequence, and the rule for its *N*th term can be written as:

$$f_1 = f_2 = 1, \; f_n = f_{n-2} + f_{n-1} \text{ for } n \geq 3$$

To put it in words, each term after the second is the sum of the two terms preceding it. Now, what's beautiful about such a series of numbers? It turns out we can use the series of numbers to calculate Phi. We can do this by dividing a Fibonacci number, let's call it f_n , by the number preceding it, f_{n-1}. This will result in a sequence that approaches Phi. The larger the terms used, the closer to Phi we get. [**3**, p.17]

$$\frac{f_5}{f_4} = \frac{5}{3} = 1.66667 \ldots$$
$$\frac{f_{10}}{f_9} = \frac{55}{34} = 1.61764 \ldots$$
$$\frac{f_{15}}{f_{14}} = \frac{610}{377} = 1.61803 \ldots$$
$$\frac{f_n}{f_{n-1}} = \phi = 1.6180339887 \ldots$$

For a more detailed view on proving that the sequence of the ratios of the Fibonacci terms converges to the Golden Ratio, see [**1,** pp. 289-292] and for proof that this is true [**3**, pp. 17-18].

The Fibonacci sequence also appears in the golden spiral, which is also known as the logarithmic spiral. The logarithmic spiral is a spiral whose polar equation is: $r = a \cdot e^{b\theta}$, where r is the distance from the origin, θ is the angle from the x-axis, and a and b are arbitrary constants. This spiral is related to almost all of the topics discussed in this paper: the Fibonacci sequence, the golden ratio, and the golden rectangle. [**13**]

The golden rectangle is also closely related to the golden spiral and the Fibonacci Sequence. A rectangle is considered golden if the ratio of its length to its width is equal to ϕ. In other words, the rectangle's sides would have a ratio of 1: ϕ. The golden ratio is defined such that partitioning the original rectangle into a square, with the new rectangle that is formed also having the ratio of its sides equal 1: ϕ. [**12**]

As you can see in figure 2, the sequence and its terms can be directly placed into the areas of both the spiral and rectangle, resulting in a golden spiral that also involves the sequence itself that is bordered by a golden rectangle. Its beauty comes from the fact that its form remains relatively the same even as it grows in size. When the length of the spiral increases, the radius also increases proportionately, and so the actual shape/form of the spiral remains the same. [**3**, p. 20] For this reason, it is also known as the equiangular spiral due to the nature of its intersection angles. It has been shown that the acute angle formed between any radial vector to a point on the curve and the tangent line to the curve at that point remains the same for all values of θ, which is roughly around 73° [**5**]. The spiral continues inwards and outwards indefinitely and approaches a point known as a center. You can find the center of this spiral by finding the intersection points of the diagonal lines BE and AC as shown in the picture.

Now that we've established the basics of the golden ratio, the golden rectangle, and the Fibonacci sequence, we can further discuss the beauty and "divinity" that surrounds it. There is not much spectacular about the golden ratio itself (unless you consider φ' being its own negative reciprocal spectacular). It is only when you apply it to architecture, or visual arts does its beauty shine. From Pythagoras's days until modern times, the golden ratio can be seen throughout

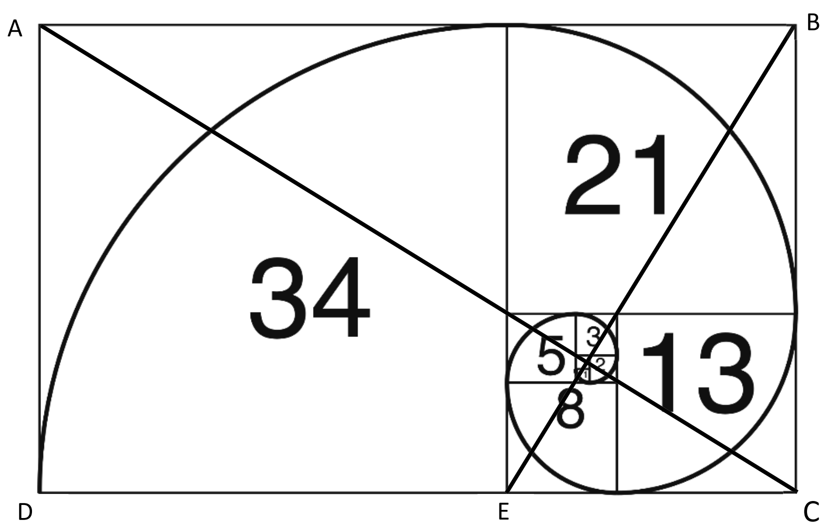


*Figure SEQ Figure * ARABIC 2 Fibonacci Sequence in the Golden Spiral bordered by a Golden Rectangle*

nature and the art humans have produced. Some well-known examples of the artistic rendition of the golden ratio include Leonardo da Vinci's *Vitruvian Man*, *The Last Supper*, and Michelangelo's *The Creation of Adam* (images included at the end) The dimensions in the *Vitruvian Man* include the golden ratio. The measurements of the man's arm length and height, which correspond to the square and circle, respectively, are in a golden ratio. In Michelangelo's *Creation of Adam*, the point where the fingers of God and Adam touch is found to be at the golden ratio of both their horizontal and vertical dimensions. [**2**, p. 704]

Leonardo da Vinci's *Mona Lisa* is another famous painting brought up whenever the golden ratio and visual arts are discussed. It is by far one of da Vinci's most popular pieces and arguably the most controversial when discussing its golden ratio implementation. As Meisner states in his work, you can find multiple evaluations of the golden ratios, spirals, and rectangles, and there's no possibility that all of them are correct. However, using the measurements of the canvas itself to base the golden ratio on, we can see that there is still evidence of its appearance (shown in the images at the end) [**8**, p.68] We can see that the top of her head is clearly bound by the golden rectangles, while the sides of her hair are approximately bound as well. Furthermore, we can see that her left eye is almost exactly in the center of the frame. Her left and right arms stay mainly on the left and right rectangles, while her chest remains centered.

It is unsure how much of this was actually intended. However, it seems that for the most part, Leonardo da Vinci's and Michelangelo's applications of golden rectangles and the golden ratio were by no means accidental. Based on the detailed inspections of the proportions and sections of the aforementioned artists by Gary B. Meisner, it seems very plausible to assume that the golden proportions, rectangles, and ratios were considered in many drawings from the Renaissance era, though we still do not know for certain. [**8**, pp. 66-74 and 84-89]

The golden ratio and spiral can also be found as a reoccurring natural phenomena. The reasoning behind the "divinity" of the golden ratio can be argued behind its common occurrence in nature. One could question why Mother Nature would design anything less than efficient and "perfect" that could be found so commonly spread around our world. Seen in the formation of animal and seashells, the arrangement of petals on flower buds, and in animal patterns such as butterfly wings, the golden ratio can be seen everywhere if one looks hard enough for it. The beauty of the golden ratio comes not only from mathematics, but from the fact that it can be found in almost everything, both natural and artificial. A ratio so standard that it elegantly tells us the world we live in is built on mathematics and patterns. To lead off with a quote by Mario Livio:

> Some of the greatest mathematical minds of all ages… have spent endless hours over this simple ratio and its properties. But the fascination with the Golden Ratio is not confined just to mathematicians. Biologists, artists, musicians, historians, architects, psychologists, and even mystics have pondered and debated the basis of its ubiquity and appeal. In fact, it is probably fair to say that the Golden Ratio has inspired thinkers of all disciplines like no other number in the history of mathematics. [**6**, p. 6]

Mario describes the golden ratio perfectly. While it is a mathematical concept, its applications and effects range out to almost everything.

Returning to man-made structures, architecture is another big subject where you can physically see the applications of the golden ratio and the Fibonacci sequence. The literal combination of "math and art," or to put it more realistically, art that cannot exist without math, has much room for mathematical expression. The Great Pyramids of Giza are commonly the first pieces of ancient architecture that have boggled mathematicians, historians, and Egyptologists for centuries; with the immense size and weight of the limestone blocks used to build it, it is only fair to question the techniques and technologies used in the primitive days of the ancient Egyptians.

However, focusing on the architectural value behind the Great Pyramid of Khufu, we see that there is a lot more than what meets the eye. Herodotus, an ancient Greek writer and geographer, learned from the ancient Egyptian priests that the square of the Pyramid's height is equivalent to the area of its triangular lateral side.

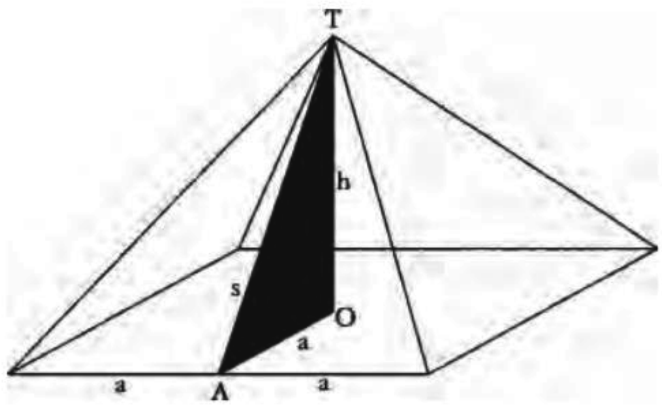


*Figure SEQ Figure * ARABIC 3 Measurements of the Great Pyramid of Giza* [**7**, p. 56]

While this statement might not have been of much use during its time, it essentially tells us that the Pyramid was explicitly built so that the ratio of the height of its triangular face to half the side of the base is equal to the golden ratio. [**6**, p. 56]

Using the figure above, we can see that a is one half of the length of the base, and s is the height of the triangular face. To quote Livio, "If the statement attributed to Herodotus is correct, this would mean that h^2 (the square of the Pyramid's height) is equal to $s \cdot a$ (the area of the triangular face). Some elementary geometry shows that this equality means that the ratio $\frac{s}{a}$ is precisely equal to the Golden Ratio." While it might not be exact, as the sides of the Pyramid are not all exactly equal in length, it still comes very close to the golden ratio. The error margin is relatively small and could be negligible. In *The Golden Ratio: The Divine Beauty of Mathematics*, Gary B. Meisner states that a pyramid based on the number Phi only varies by 0.07% from the Great Pyramid's estimated dimensions. [**8**, p. 95] Using the average measurements of the Great Pyramid of Khufu, we get the calculations:

$$2a = 755.79 \text{ feet}$$
$$h = 481.4 \text{ feet}$$
$$s = \sqrt{h^2 - a^2} = 612.01 \text{ feet}$$
$$\frac{s}{a} = \frac{612.01}{\frac{1}{2}(755.79)} = 1.6195 \approx \varphi$$

As mentioned before, we still would not know if this was intentional or purely coincidence. Livio states, "just from the dimensions of the Great Pyramid alone, it would be impossible to determine whether phi or pi, if either, was a factor in the pyramid's design." [**6**, p.59] However, one can lean more towards believing it was not intentional. According to statements made by Dr. Charlie Smith, a professor of mathematics at Park University with a Ph.D. in mathematics, the ancient Egyptian scrolls which have survived to this day contain zero evidence of knowledge of the golden ratio.

However, using [**8**, pp. 93-102], we can see that the argument that the ancient Egyptians used the golden ratio by pure chance becomes less and less valid. The golden ratio not only appears in the Great Pyramid of Khufu itself, but also in the building site of the Pyramids, at the apexes of all three of the Pyramids, and even in the Sphinx. With the consistent repetition of the golden ratio found in the Pyramid complex, it becomes very difficult to argue against the ancient Egyptian's implementation of the golden ratio in their architecture.

Another famous piece of ancient architecture we must mention would be the Parthenon. Indeed, Greek architects carried out the building of the Parthenon with the golden ratio in mind. The proportions of the Parthenon's exterior and floor plan meet the golden ratio on numerous occasions. It was a sacred temple dedicated to the Goddess Athena and was built on the Acropolis of Athens. Livio states that Adolph Zeising, the author of *"Der Goldne Schnitt"* a book on the golden section published in 1844, believes that the height of the façade from the top of its tympanum to the bottom pedestal below the columns is divided in the golden ratio by the top of the columns. [**6**, p. 73] However, once again, it is not certain whether the use of the golden ratio by the ancient Greek architects was purposeful. Mathematicians such as George Markowsky say that the information presented in articles about architecture, art, and other applications represents false or misleading formation. [**7**, p. 2] However, other mathematicians, aptly named "golden ratio enthusiasts" by Markowsky, state otherwise. In support of these enthusiasts, the current day Parthenon has partially collapsed, and so its features and dimensions might not be as accurate as they were during its construction. As such, the argument of the golden ratio being used could still very much be valid. According to Livio:

> Whether or not the golden ratio features in the Parthenon, what is clear is that whichever mathematical "programs" concerning the Golden Ratio were instituted by the Greeks in the fourth century B.C.E, that work culminated in the publication of Euclid's Elements, in around 300 B.C.E. Indeed, from a perspective of logic and rigor, the Elements was long thought to be an apotheosis of certainty in human knowledge. [**6**, p. 75]

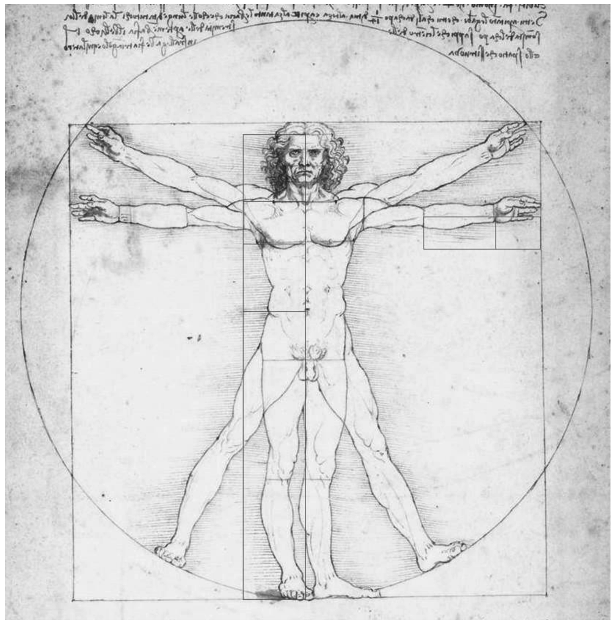
Meisner also has a detailed and beautifully done analysis of the Parthenon in his book cited [**8**, pp. 103-108].

For the third example of man-made architecture, I'd like to cover something a little more recent than the previous two: the Notre Dame Cathedral. The Notre-Dame de Paris began construction in 1163 AD and was not completed until 1260 AD. The cathedral is one of the most widely recognized buildings in France. It is a significant tourist attraction, and reasonably so, as it's one of the most beautifully designed (both mathematically and aesthetically) cathedrals of its time.

The cathedral can be split into different sections, such as the western and eastern façade and the areas of different gothic stained-glass windows. The golden ratio can be found in both the dimensions of these windows and in the western façade, which is also the main entrance into the cathedral. [**8**, p. 111] The exact numbers might be slightly skewed due to the large margin of error while measuring Notre Dame. However, compared to the Parthenon and the Great Pyramids, it is believed that the cathedral's proportions were built based on the golden ratio. So, this tells us that its inclusion was intentional and not pure coincidence.

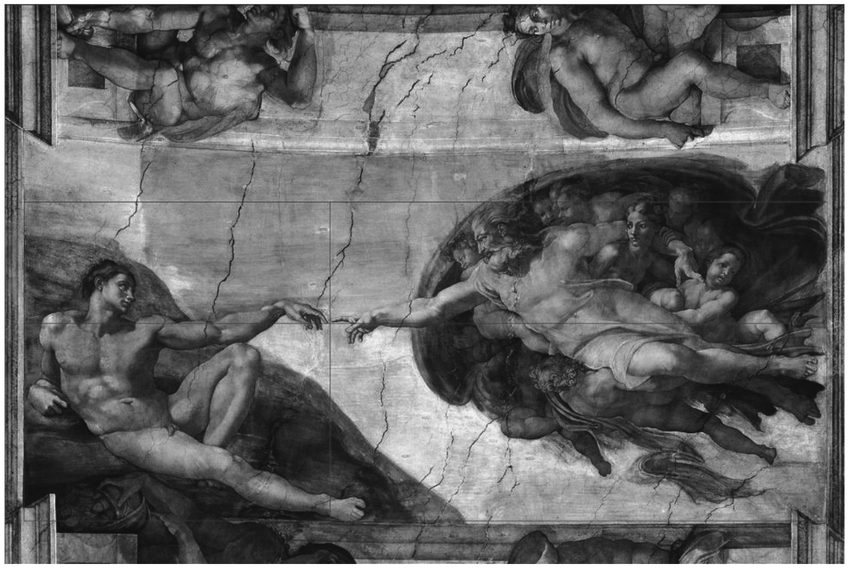
The golden ratio has been studied and analyzed by many mathematicians, historians, architects, artists, and others in the past, and is still a topic of great importance in the current day. Whether or not the intentions and the outcomes of such architects in the past, present, or future were obvious, the fact that the golden ratio can still be seen in such artworks is a testament of its beauty. As previously mentioned, the definition of "art" can be a little vague and different from one person's perspective to another, and so I thought it would be fitting if instead of focusing on art formed by mathematics, I'd focus on the physical applications of mathematics in the real world, both natural and man-made. As such, I would be capturing the beauty of the math while still maintaining the focus on the objects in question.

Somewhat of a recurring theme was evident in almost every real-life application I had presented: We don't know for sure that the artists used the golden ratio intentionally. While it does leave the position open for debate, it should not be mistaken as evidence that its occurrence is merely coincidence. The artistic nature of humans is intricate and complicated to determine. As most of these pieces are hundreds to thousands of years old, it's impossible to say what the creator was thinking during the creation process. However, one thing is certain: The beauty of the underlying mathematics behind all these structures and paintings remain as beautiful and aesthetically pleasing as initially intended.

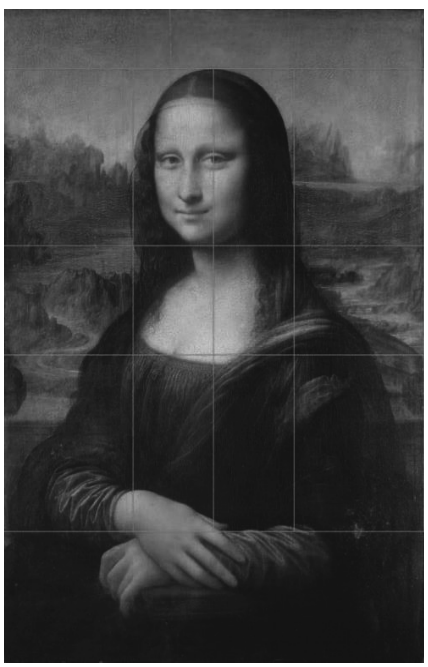


Vitruvian Man, Leonardo da Vinci (1490)

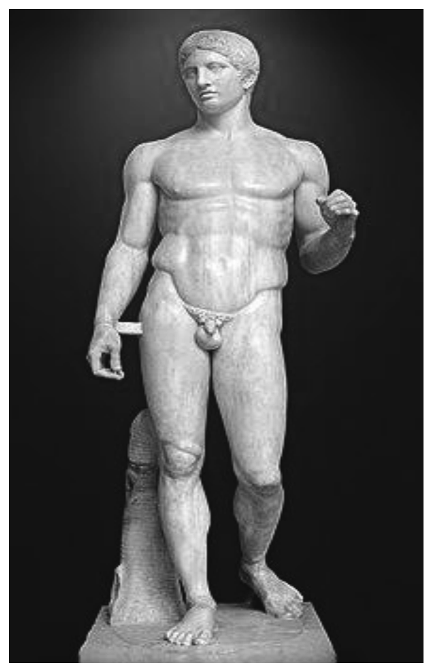
Last Supper, Leonardo Da Vinci (1495-1498)



The Creation of Adam, Michelangelo (1512)



Mona Lisa, Leonardo da Vinci (1503)



A replica of Doryphoros of Polykleitos

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